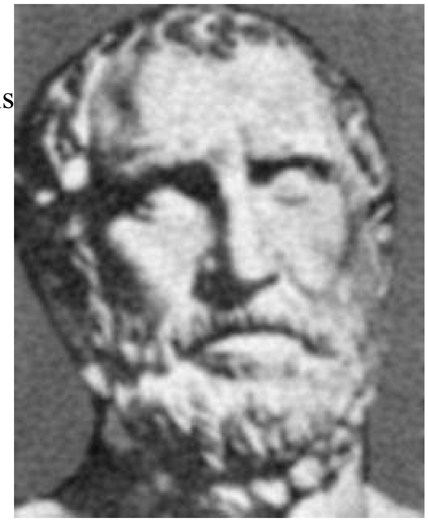
Zeno



Greek philosopher and mathematician **Zeno of Elea** (c. 490 BCE – c. 425 BCE) was the first great philosophical skeptic. He is famous for his paradoxes, which deal with the continuity of motion. He made a series of arguments in which he purported to prove by logical means that motion and plurality are impossible. In his view all human knowledge is based on an unprovable hypothesis: that time and space are continuous. In *A History of Mathematics*, Carl B. Boyer wrote:

> "The arguments of Zeno seem to have had a profound influence on the development of Greek mathematics, comparable to that of the discovery of the incommensurable, with which it may have been related…. The realm of number continued to have the property of discreteness, but the world of continuous magnitudes (…) was a thing apart from number and had to be treated through geometric method. It seemed to be geometry rather than number that ruled the world. This was perhaps the most far-reaching conclusion of the Heroic Age, and it is not unlikely that this was due in large measure to Zeno of Elea and Hippasus of Metapontum."

Zeno was born and died in Elea, which is in southern Italy, and he is identified by his place of birth to distinguish him from other Zenos of antiquity. According to E.T. Bell (*The Development of Mathematics*), by tradition Zeno of Elea was:

> "a pugnacious dialectician with a passion for being different from everyone else….And it is told that his uncompromising intellectual honesty finally cost him his life. He had conspired with the political faction which lost, and met his death by torture with heroic fortitude."

According to Plato, Zeno's now lost treatise, in which he indirectly argued against the reality of multiplicity and of motion, consisted of several discourses. In each, Zeno made a supposition and then gave an argument that led to an absurd consequence. He may have been the first to employ this method of indirect proof, now referred to as *reductio ad absurdum.* Bell wrote that the paradoxes should be called "sophistries," which are logical arguments that some cannot accept but neither can they refute them. Aristotle argued against Zeno's paradoxes, calling them "fallacies," without being able to show where they were false.

Originally there were forty paradoxes, but only eight have survived. Zeno's arguments concerning motion introduced the element of time, and revealed that time cannot be considered merely the sum of moments. Briefly the four arguments are as follows:

1. *The Dichotomy*: Motion cannot exist because if something moves from one place to another, it must first reach the midpoint of the distance to be traveled, but before it can do that it has to reach the midpoint of the first half, and before it can do that it must reach the midpoint of the first fourth, and so on ad infinitum. It must, therefore, pass through an infinite number of points, and this is impossible in a finite amount of time.

2. *The Achilles*: In a race between the running Achilles and the crawling tortoise, the former can never overtake the latter if the tortoise has a head start. Before Achilles reaches the point from which the tortoise started, the tortoise will have move ahead a little way and Achilles must run to this new position but by the time he reaches it the tortoise has moved ahead again, and ad infinitum. English mathematician and writer Charles Dodgson, better known as Lewis Carroll, used the characters of Achilles and the tortoise to illustrate his paradox of infinity.

3. *The Arrow*: An arrow shot in the air is either in motion or at rest. An arrow cannot move, because for motion to take place, the arrow would have to be in one position at the beginning of an instant and at another at the end of the instant. But as time is made up of instants, which are time's smallest measure and are not further divisible, this is a contradiction. Hence, the arrow is always at rest.

4. *The Stadium*: This is the most awkward of Zeno's paradoxes to describe. It concludes that half the time is equal to twice the time. Suppose that there are three rows of soldiers, A, B, and C [Figure 7.4].

First Position							Second Position			
A		*	*	*			A	*	*	*
B	*	*	*				B	*	*	*
C			*	*	*		C	*	*	*

Figure 7.4

In the smallest unit of time A remains stationary while rows B and C move at equal speeds in opposite directions. When they have reached the second position, each B has passed twice as many C's as A's. Then it takes row B twice as long to pass row A as it does to pass row C. However, the time for rows B & C to reach the position of row A is the same. Thus half the time is the same as twice the time. The paradox arises from the assumption that space and time can be divided only by a definite amount.

Mathematicians, physicists, and philosophers came to realize that to escape the contradictions found in Zeno's paradoxes, it was necessary to radically reinterpret the concepts of space, time, and motion, as well as the mathematical ideas of line, number, measure, and the sum of a series. Zeno's integers had to be replaced with modern notions of real numbers. Contributors to the resolution of the paradoxes include: Isaac Newton, Gottfried von Leibniz, Augustin Cauchy, Karl Weierstrass, Richard Dedekind, Georg Cantor, Albert Einstein, and Henri Lebesgue. In *Reflections on Relativity*, Kevin Brown wrote:

> "Surely we can forgive Zeno for not seeing that his arguments can only be satisfactorily answered – from the standpoint of physics – by assuming Lorentzian invariance and the relativity of space and time."

Zeno also made arguments against multiplicity, showing that the continuous cannot be composed of units no matter how many or how small. He had two principal arguments.

1. If it is assumed that a line segment is composed of a multiplicity of points, then the segment can be bisected. Then the first half of the segment is itself a segment that can be bisected, and the first half of this segment is a segment that can be bisected. Continuing this bisection process we will never come to a point, a stopping place, so the original line segment cannot be composed of points.

2. The line must be both limited and unlimited in the number of its points. It is limited since it contains just as many points as compose it, no more, no less. It therefore must have a definite number of points, and a definite number is a finite or limited number. However, the line must also have an unlimited number of points, for it is infinitely divisible. Therefore supposing a line has a multiplicity of points leads to a contradiction.

Quotation of the Day: “If there are many, they must be just as many as they are, neither more nor fewer. But if they are just so many as they are, they must be finite [in number]. If there are many, the existents are infinite [in number] between existents, and again others between these. And thus the existents are infinite [in number]” – Zeno of Elea