Leonardo Pisano (Fibonacci)

Called by many the "greatest European mathematician of the middle ages," **Leonardo Pisano** was generally known as **Leonardo of Pisa** (1170 – c. 1250) for his place of birth or by his nickname **Fibonacci** (i.e. filius Bonaccii). There is no record of exactly when he died or of what cause, nor is there any information about his private life. His merchant father, Guilielmo Bonaccii, was the public notary at the customhouse at Bugia (now Bejaia), a Moorish Mediterranean port in North Africa. It was through his extensive travels with his father that Leonardo got his education. He was one of the earliest-known Europeans to learn of the Hindu-Arabic or decimal numeration system as well as of Al-Khwarizimi's Algebra, the *al jabr wa'l muquabalah*.

Leonardo returned to Italy about 1200, where he published his best-known work, the *Algebra et almuchabala* (1202), popularly known as the *Liber abbaci* ("Book of the Abacus" or "Book of Calculating"). In it he explained the Arabic system of numeration, illustrating its advantages over the Roman system still in use in Europe. The *Liber abbaci* contained an account of algebra, with Leonardo showing the convenience of using geometry to get rigorous demonstrations of algebraic formulas. He showed how to solve both determinate and indeterminate equations of the first and second degree. The book was a huge success and remained a standard source for mathematical writers for several centuries. He claimed he wrote the book, "in order that the Latin race might no longer be deficient in that knowledge [the Hindu-Arabic numerals]."

As the *Liber abbaci* predated the invention of the printing press, it originally circulated through hand-copied manuscripts only. Because so few copies were made, it is fortunate that one still survives. Also extant is Leonardo's *Practica geometriae* (1220, "The Practice of Geometry"), which includes some propositions and problems of trigonometry, and his *Liber quadratorum* (1225, "The Book of Squares"), his most original work, whose subject was number theory. It was almost entirely ignored during the Middle Ages, only to be rediscovered some three hundred years later. Unfortunately, his commercial arithmetic *Di minor guisa* is lost as is his commentary on Euclid's *Elements*.

Fibonacci is best remembered for a problem he introduced in the *Liber abbaci*. "A certain man put a pair of rabbits in a field. If rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?" Assuming none of the rabbits die or escape this problem leads to the sequence of numbers today known as the *Fibonacci sequence*: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,144... (1). The first 1 represents the original pair of rabbits, which is still the only pair after the second month, there are 2 pairs after the third month, 3 after the fourth, and the number for each succeeding month is the sum of those of the two previous months' pairs.

Whereas rabbits may not behave as in Fibonacci's problem, his numbers are often embodied in nature. The arrangements of the seeds in a sunflower are found in small diamond-shaped pockets bound by spiraling curves radiating outwards from the center of the head both to the left and the right. The number of clockwise spirals and the number of counterclockwise spirals are successive terms in the Fibonacci sequence. This arrangement appears to keep the seeds uniformly packed no matter how large the seed head. This is true of any composite flower, such as a daisy or an aster. The sequence also can be noted in the leaves, buds or branches growing out of the side of a stalk of a plant, leaves of a head of lettuce, layers of an onion, and conical spirals of a pinecone.

The significance of his problem seems to have escaped Fibonacci, as he did not seem to explore its uses and properties. Could he have been aware of the relation of his numbers to the so-called *golden ratio*? The ancient Greeks stated that a line segment *AC* is divided into the *golden ratio* by the point *B* if *AB*/*BC* = *BC*/*AC*. This ratio, represented by the Greek letter *ø* (phi), is equal to $(\sqrt{5} - 1)/2 = 0.61803$...and 1 + *ø* (or 1/*ø*) is called Phi, which equals $(\sqrt{5} + 1)/2 = 1.618$… The *golden ratio* and the golden rectangle are found in Greek architecture and pottery. They have been applied to sculpture, painting, architectural design, furniture design and type display. If the sequence of ratios of successive terms of the Fibonacci sequence is formed, we have 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, 55/89, whose limit is the *golden ratio*.

An interesting pattern is found by examining the units digit of the numbers in the sequence (1), namely: 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, … (2). The pattern may not be apparent from these few terms, but if is carried out to 60 places, it will be noted that this same pattern of digits repeats again and again all the way through the Fibonacci series. This property is described by saying that the series of final digits of Fibonacci numbers repeats *with a cycle of 60*. For the patient, an examination of the sequence of the last two digits of Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 44, 33, 77, 10, 87…(3) has a repeating cycle of 300 digits, that is, after 300 numbers in the sequence, the pattern of numbers repeats throughout the Fibonacci series. Those who care to explore further will find that the sequence of the last three digits of Fibonacci numbers has a repeating cycle of 1500 numbers; the sequence of the last four digits of Fibonacci numbers has a repeating cycle of 15,000 numbers.

The following notation is used to refer to the various terms of the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, and so on. The Fibonacci sequence is characterized by the recursion formula: $F_{n+1} = F_n + F_{n-1}$ (4). Number sequences satisfying the recursion formula (4), starting with any two integers have also been named Fibonacci sequences and have been studied. For instance, the sequence beginning with the numbers 2 and 5 is: 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, … but the two sequences with the most distinctive properties are the original Fibonacci sequence and the Lucas sequence, named for French mathematician Edouard Lucas (1842 – 1891), who was the one that introduced the name Fibonacci numbers. The *Lucas numbers* are given in the following sequence: 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, … (5).

For the Lucas sequence (5), the following notation is employed: $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, and so on. The Lucas sequence is characterized by the recursion formula $L_n = L_{n-1} + L_{n-2}$ for $n > 1$. The Lucas numbers have many properties similar to the Fibonacci numbers and a number of interesting relations exist between the two sequences. For instance, if each Fibonacci number is multiplied by its corresponding Lucas number, we find the following correspondence:

n:	0,	1,	2,	3,	4,	5,	6,	7,	8, ….
F_nL_n:	0,	1,	3,	8,	21 ,	55,	144,	377,	987, …

To see the relation, consider the sequence of Fibonacci numbers with even subscripts:

n:	0,	1,	2,	3,	4,	5,	6,	7,	8, …
F_{2n}:	0,	1,	3,	8,	21,	55,	144,	377,	987, …

It has been proved that $F_nL_n = F_{2n}$. *The Fibonacci Quarterly* is a journal devoted principally to research on the Fibonacci sequence. In has been in existence since 1963 and is still going strong.

Quotation of the Day: "… considering both the originality and power of his methods, and the importance of his results, we are abundantly justified in ranking Leonardo of Pisa as the greatest genius in the field of number theory who appeared between the time of Diophantus and that of Fermat. – R.B. McClenon