Menaechmus

Allegedly **Menaechmus** (c. 380 BCE – c. 320 BCE) was the first to investigate curves that would come to be known as the ellipse, the parabola, and the hyperbola as sections of a cone. For a long time these curves were called the *Menaechmian triads*. A brother of the famous geometer Dinostratus, he was born at Alopeconnesus in Thrace (now in Turkey). According to Proculus he was a friend of Plato and a student of Eudoxus, whom he probably succeeded as head of the school at Cyzicus. Such was his fame as a teacher of geometry that he was appointed a tutor of Alexander the Great. Menaechmus wrote a number of treatises that have not survived except in small fragments.

Menaechmus discovered the conic sections as a by-product of his attempt to solve the "Delian problem," that is, the duplication of the cube. The problem was reportedly inspired by a plague that killed about a quarter of the Athenian population. A delegation of Athenians approached the oracle of Apollo at Delos to ask how the plague could be ended. The answer was that the altar of Apollo had to be doubled. The Athenians doubled the dimension of the altar, but this wasn't the answer because it increased the altar eightfold in volume. This was the origin of the problem that may be stated as follows: "given the edge of a cube, construct with compass and straightedge alone the edge of a second cube having double the volume of the original one." More than two millennia would pass before it was finally demonstrated that the task is impossible. In the meantime, a great deal of useful mathematics was inspired by the problem. What is known of Menaechmus' application of the conic sections to the solution it is found in Eutocius' anthology *Commentary on Archimedes' Sphere and Cylinder* (6th century CE). Menaechmus' solution of the problem made use of more than the specified instruments, causing Plato's disapproval because he hadn't played by the "rules," having used "mechanical" devices.

In modern notation and language the problem is this: given a line segment of length a, we must find a line segment of length x such that $x^3 = 2a^3$. Hippocrates of Chios (fl. Ca. 430 BCE) showed that this could be achieved provided it was possible to find two mean proportionals x and y: i.e., for a given line segment of length a it is necessary to find line segments of length x and y such that

$$a/x = x/y = y/2$$

for in that case $a^3/x^3 = (a/x)^3 = (a/x)(x/y)(y/2a) = a/2a = 1/2$. At the time, the Greeks had only two means of discovering new curves. The first was through a sequence of uniform motions and the second through the intersection of known geometric surfaces. Menaechmus described a conic section to be a curve bounding a section of a right circular cone cut off by a plane perpendicular to the cone's surface. The *parabola* was obtained by intersecting a plane with a right-angle cone (the vertex angle is a right angle), the *hyperbola*, with an obtuse-angled cone (the vertex angle is an obtuse angle), and the *ellipse*, with an acute-angled cone (the vertex angle is an acute angle). He didn't use the familiar names for the conic section, which Apollonius coined much later. Menaechmus used the descriptive words *orthotome,* "right cut" for a parabola; *amblytome*, "dull cut" for a hyperbola; and *oxytome,* "sharp cut" for an ellipse.

Menaechmus developed two methods for showing how these curves could be used to obtain a solution of the duplication of a cube problem. To state the case simply: if x and y are the required two mean proportionals between two line segments of lengths a and b, that is,

$$\text{if } a/x = x/y = y/b \text{ then } y = x^2/a,\ x = y^2/b, \text{ and } xy = ab.$$

The first two equations represent parabolas and the third a hyperbola. Menaechmus's first solution used the intersection of the second and third of these conics [Figure 10.1], that is a parabola and a hyperbola, whereas, his second used the intersection of the first and second [Figure 10.2], that is, two parabolas.

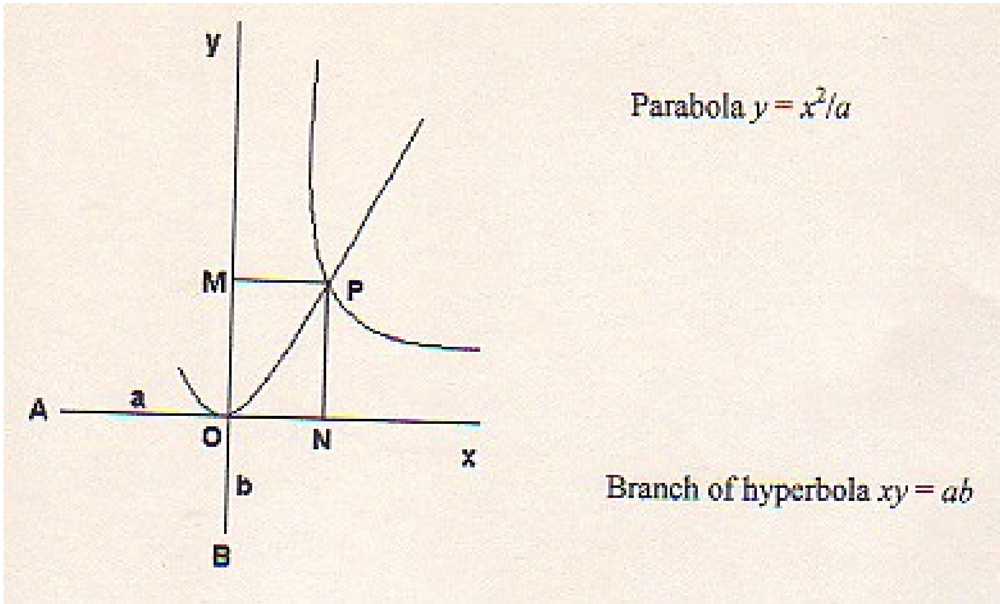

Figure 10.1

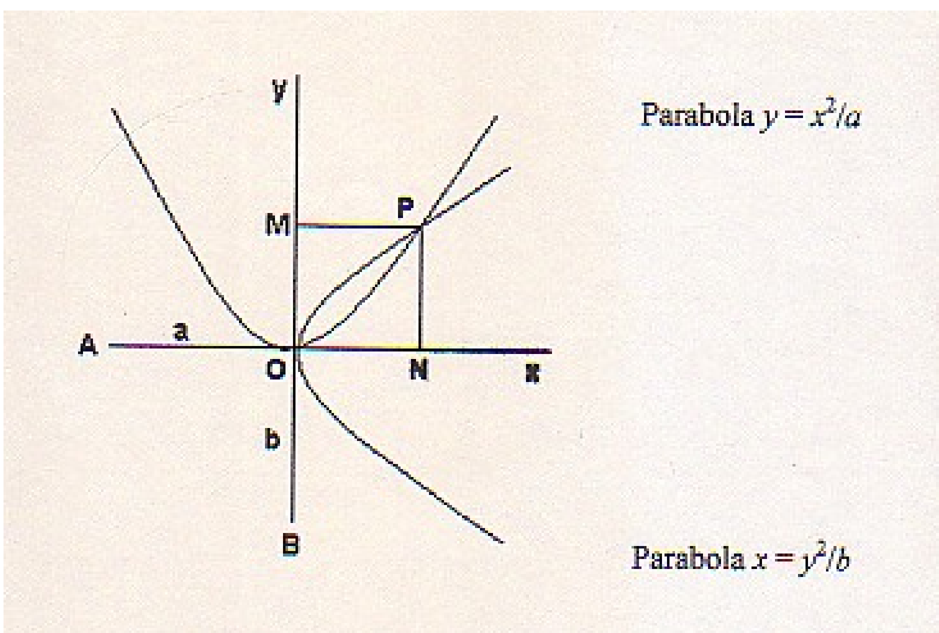

Figure 10.2

Recorded history tells us little about Menaechmus' other accomplishments as a geometer. Proculus said that he was a supporter of Eudoxus's theory of heavenly bodies based on concentric spheres, but that he postulated a larger number of spheres. He made no distinction between theorems and problems; to him they were all problems, even if the one required finding a proof and the other finding a solution. Stobaeus related a familiar anecdote about Menaechmus, which other authors attributed to Plato. Alexander, probably in a hurry to get out in the field and conquer the world, asked his master to show him an easier way to learn geometry. Menaechmus is alleged to have replied: "O king, for traveling through the country there are private roads and royal roads, but in geometry there is one road for all."

Some scholars believe Menaechmus may have had recourse to some rudimentary notions of analytic geometry. However, it is unlikely that he had any idea of a relation between curves and equations. There is no evidence that at his time there was any symbolic way of representing unknowns except geometrically. It's of no great importance to speculate on what inklings he had of mathematical techniques that wouldn't be developed for another 2000 years. It's enough for their role in the further development of mathematics that he was inspired to find the curves he did. He did not study conics with any particular application in mind. It was a theoretical exercise and the most significant application had to wait some eighteen centuries for Johannes Kepler to find that the orbits of planets were ellipses.

Quotation of the Day: "If the Greeks had not cultivated Conic Sections, Kepler could not have superseded Ptolemy; if the Greeks had cultivated Dynamics, Kepler might have anticipated Newton." – William Whewell