Henri Lebesgue

French mathematician Henri Léon Lebesgue (June 28, 1875 – July 26, 1941) launched the modern theory of functions of a real variable. His work that of Emile Borel, who created the first effective theory of the measure of sets of points and René-Louis Baire, who set up a classification of functions. Intrigued by problems associated with Riemann integration, Lebesgue developed new theories of measure and integration that bear his name. By generalizing the ideas of length and area to a general concept of the measure of a set, he allowed an integral to be assigned to a much broader class of functions. Integrals and derivatives were known before the time of Newton and Leibniz, who showed that areas of a region under a curve could be determined by finding a function whose derivative is the function representing the curve. Later Cauchy investigated the integrals of continuous functions. Riemann refined Cauchy's definition of integral and extended it to the integrals of certain discontinuous functions. Lebesgue's definition of integral allowed more integrable functions than Riemann's.

Lebesgue was born in Beauvais, Oise, Picardie, France. His father and his two sisters each died of tuberculosis shortly after his birth. He also contracted the disease and suffered from its aftereffects throughout his life, with his health remaining always fragile. His mother was an untiring worker, who proudly supported her son's education. He had the usual mathematical training, but showed exceptional irreverence in questioning his professors at the École Normale Supérieure. After graduation he taught for a few years in preparatory schools while writing his doctoral dissertation, *Intégrale, longueur, aire*, which was accepted at the University of Nancy in 1902. In it he developed his theory of measure and integration that revolutionized analysis. He was appointed *maître de conférences* (lecture master) at the University of Rennes, and then in 1906, he moved to Poitiers, first as *chargé de cours* (assistant lecturer) and later professor. In 1912 he was appointed at the University of Paris and afterwards became professor at the Collège de France, where he was a superb teacher. In addition to his work on measure theory and integration, Lebesgue contributed to topology, Fourier analysis, and potential theory. In 1917 he was awarded the Prix Saintour and five years later was elected to the Paris Academy of Sciences. Lebesgue died on July 26, 1941 in Paris.

Lebesgue's theory is based on the notion of measure of sets of points, its generality resting upon the fact that a Lebesgue integrable function, unlike a Riemann integrable function, need not be "continuous almost everywhere" (that is, except on a set of measure zero). In the Riemann case, the area under the graph of a function, $y = f(x)$, defined on a closed interval [a, b], is found by partitioning the domain of the function along the x-axis and finding the sum of the areas of vertical rectangles [a Riemann sum] that approximates the area of the region bounded by the function, the x-axis, and the lines $x = a$ and $x = b$ [Figure 6.15]. For functions that are continuous almost everywhere, finer partitions of the domain, result in better approximation of areas of the region, with the Riemann integral being defined as the limit of the sequence of approximations. Unfortunately, some functions do not have well-defined limits, and so have no Riemann integral.

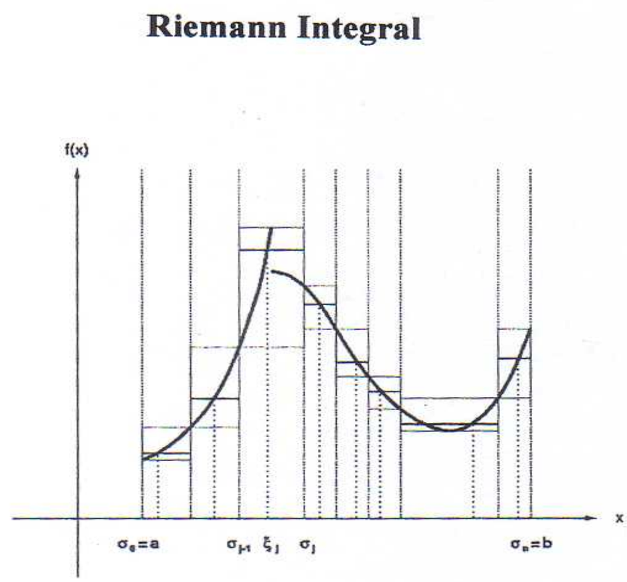

Figure 6.15

In the case of the Lebesgue integral, rather than partition the domain along the *x*-axis, the range along the *y*-axis is partitioned into intervals [Figure 6.15]. Instead of using areas of rectangles, Lebesgue built up his integral for what he called simple functions (for instance, a step function) that take on only finite many values. Then he defined the integral for more complicated functions as the upper bound of all the integrals of simple functions that were less than equal too the function in question on the same intervals [Figure 6.16]. As part of the development of Lebesgue integration, he invented the concept of Lebesgue measure, which measures lengths rather than areas, leading to the modern theory of measure. Every function with a Riemann integral also has a Lebesque integral, and the two agree, but there are many functions with a Lebesgue integral that have no Riemann integral.

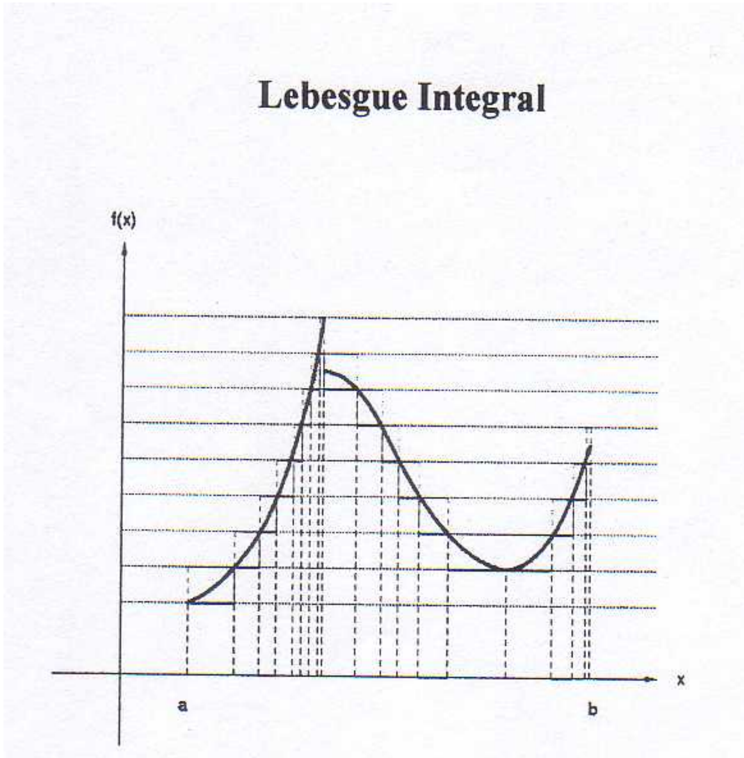

Figure 6.16

Because it is simpler and more intuitive than Lebesgue's integral, the Riemann integral suffices for most practical purposes. The more complicated Lebesgue integral is widely used by mathematicians because of its more abstract nature. In his *Leçons sur l'intégration et la recherché des fonctions primitives* (1904), Lebesgue showed that a bounded function is Riemann integrable if and only if the points of discontinuity form a set of measure zero. An example a set of measure zero is the following: From a line segment select a set of at most a countably infinite number of points such that there is always a line segment, no matter how small, between any two consecutive points. In theoretical cases the Lebesgue integral affords simplifications. It is particularly useful in the theory of Fourier series.

With his definition of the definite integral, Lebesgue was able to extend the concept of area bound by curves to include many more discontinuous functions. His work also advanced the theory of multiple integrals. His innovations, among the finest mathematical contributions of the 20th century, finally won approval, but not without some resistance. Those who objected to functions without derivatives were equally dismissive of Lebesgue's non-differentiable surfaces. Applied mathematician Richard Hamming remarked: "If the prediction that an airplane can stay up depends upon the difference between Riemann and Lebesgue integration, I don't want to fly in it."

The Lebesgue integral has a number of shortcomings. There are functions whose improper Riemann integrals exist but are not Lebesgue-integrable. In 1957 Jaroslav Kurzweil introduced an integral that can handle a larger class of functions than either the Riemann integral or the Lebesgue integral. Ralph Henstock further developed the theory and his definition of an integral is an even more general notion, based on Riemann's theory rather than Lebesgue's. It includes both the Lebesgue integral and improper Riemann integrals. The Henstock-Kurzweil formulation, the so-called *gauge integral*, is only slightly different from the Riemann integral and considerably simpler than the Lebesgue integral. But it is not the perfect concept either. It depends on specific features of the real line and doesn't generalize as well as the Lebesgue integral. No doubt more generalizations of existing definitions of integrals will occur over time.

Quotation of the Day: "The only instruction which a professor can give, in my opinion, is to think in front of his students." – Henri Lebesgue